Three-Space Property on Normed Spaces

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Outline of the talk

Let M be a closed subspace of a Banach space X.

A property P is a **three-space property** if two of the spaces X, M, X/M have the property P, then the third must also have the property P.

We shall discuss the following in the lecture.

- Finite-dimensionality is a three-space property.
- Completeness is a three-space property.
- Separability is a three-space property.
- Reflexivity is a three-space property.
- Dunford-Pettis property is not a three-space property.

Quotient space

Let M be a subspace of a vector space X.

We can define a new vector space called the **quotient space**, or **factor space** whose underlying set is the collection $\{x + M : x \in X\}$ of *all translates* of M.

The translates of M are obtained by an equivalence relation (verify) defined by $x \sim y$ iff $x - y \in M$. The set of all such equivalence classes $\{x + M : x \in X\}$ will be referred to as X/M (read as X modulo M).

The translate $x + M$ is called the **coset** of M containing x.

Quotient space

We define $(x + M) + (y + M) = (x + y) + M$.

We add the particular representative of the equivalence classes and take the equivalence class to which their sum belongs.

Similarly, if $\alpha \in \mathbb{K}$, we define $\alpha(x + M) = (\alpha x) + M$.

We can show that the operations of addition and scalar multiplication do not depend on the representatives chosen.

Thus X/M is a vector space.

The zero of X/M is M and $-(x + M) = (-x) + M$.

The dimension of X/M is **codimension** of M with respect to X (or, the deficiency of M with respect to X), denoted by $\text{codim} M = \dim(X/M)$.

Examples of quotient space

Example 1.

In $\mathbb{R}^2$, $M = \{(x, y) : x = 0\}$, *the* y*-axis.*

Then $\mathbb{R}^2/M$ *is the collection of all vertical lines in the plane, with the norm of each such line being its distance from the origin, that is, the absolute value of its* x*-intercept. The coset* $(1, 2) + M$ *is the set* $\{(x, y) : x = 1\}$.

Some results are based on geometric ideas that are easier to visualize if this example is kept in mind.

Examples of quotient space

Example 2.

In $\mathbb{R}^3$, *if* $M = \{(x, y, z) : z = 0\}$ *(xy-plane), then the translate of* M *containing* $(1, 2, 3)$ *is the plane* $\{(x, y, z) : z = 3\}$ *parallel to the xy-plane. Here dim* $\mathbb{R}^3/M = 1$.

Similarly, if $M = \{(x, y, z) : x = y = 0\}$ *(z-axis), then the translate of* M *containing* $(1, 2, 3)$ *is the line* $\{(x, y, z) : x = 1$ *and* $y = 2\}$ *parallel to the z-axis. Here dim* $\mathbb{R}^3/M = 2$.

Example 3.

Consider the linear space $c^{(3)}$ *of all sequences* $x = (x_n)_{n\in\mathbb{N}}$ *such that* $(x_{3k+q})_{k=0}^{\infty}$ *converges for* $q = 0, 1, 2$ *and the subspace* c_0 *such that* $\lim_{n=1}^{\infty} x_n = 0$.

Every $x \in c^{(3)}$ *can be represented as* $x = b_1 e_1 + b_2 e_2 + b_3 e_3 + a$ *where* $e_1 = (1, 0, 0, 1, 0, 0, 1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, 0, 1, 0, 0, 1, 0, \ldots)$, $e_3 = (0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots)$ *and* $a \in c_0$.

$[e_1]$, $[e_2]$ *and* $[e_3]$ *form a basis for* $c^{(3)}/c_0$ *and dim* $c^{(3)}/c_0 = 3$.

Examples of quotient space

Example 4.

$\hat{c}$ is the linear space of double sequences $x = (x_n)_{n=-\infty}^{\infty}$ such that the limits $b_1 = \lim_{n\to\infty} x_n$ and $b_2 = \lim_{n\to-\infty} x_n$ exist and the subspace $\hat{c}_0$ such that $\lim_{n\to\pm\infty} x_n = 0$.

Every $x \in \hat{c}$ can be represented as $x = b_1 e_1 + b_2 e_2 + a$ where $e_1 = (\dots, 0, 0, 1, 1, 1, \dots)$, $e_2 = (\dots, 1, 1, 0, 0, 0, \dots)$ and $a = (a_n)_{n=-\infty}^{\infty} \in \hat{c}_0$.

$[e_1]$ and $[e_2]$ form a basis for $\hat{c}/\hat{c}_0$ and $\dim \hat{c}/\hat{c}_0 = 2$.

Norm for X/M

Having noted that X/M is a linear space with respect to the operations defined above, we now wish to suppose that X is a normed space and exhibit a norm for X/M. To do this, it is reasonable to ask if the norm of X induces a norm on X/M in some natural way. When $M \neq \{0\}$, $\|x + M\| = \|x\|$ will not be a norm on X/M.

This situation helps us to think first about distance and then recover the norm from the notion of distance. The members of X/M are subsets of X, there is a natural way to define the distance between subsets $x + M$ and $y + M$ as the distance between cosets $x + M$ and $y + M$:

$$\begin{aligned} d(x+M, y+M) &= \inf\{\|u - v\| : u \in x + M, v \in y + M\} \\ &= \inf\{\|x - v\| : v \in y + M\} = d(x, y + M). \end{aligned}$$

Quotient norm

If $x \in \overline{M} \setminus M$, then $0 \leq d(x+M, 0+M) = d(x, M) = 0$ even though $x + M \neq 0 + M$. Note that $d(x, M) = 0 \iff x \in \overline{M}$.

If the function d is to have any hope of being a metric on X/M, then the set $\overline{M} \setminus M$ must be empty; that is, the set M must be closed.

Let M be a closed subspace of a normed space X. The **quotient norm** of a coset $x + M$ can be interpreted to be distance from the point x to the set M, or as the distance from the origin of X to the set $x + M$, since $d(x, M) = d(x+M, 0+M) = d(0, x+M) = \|x + M\| = \inf \{\|x + m\| : m \in M\}$, the quotient norm is a norm of X/M.

Examples of quotient space

Example 5.

Let $X = C[0,1]$ *with sup norm. Then* $M = \{f \in X : f(0) = 0\}$ *is a closed subspace of* X*. Each coset contains a constant function.*

Suppose $f(0) = a$*, then the two cosets* $[f]$ *and* $[a]$ *are same, where* a *is a constant function which takes the value* a*. Each member* f *in the quotient space* X/M *can be identified with the scalar* a*. Hence* $\dim X/M = 1$*.*

When $\mathbb{K} = \mathbb{R}$*,* M *is all functions whose graphs passing through* $(0,0)$*. The coset* $[f]$ *where* $f(0) = a$ *is all functions whose graphs passing through* $(0,a)$*.*

Examples of quotient space

Example 6.

Let $X = C[a, b]$ with sup norm and $t_1, t_2, \ldots, t_n$ be distinct points in $[a.b]$. Let $X_n = \{x \in X : x(t_j) = 0, \ \text{for all } j = 1, 2, \ldots, n\}$. Then X_n is a closed subspace of X.

The dimension of X/X_n is n because each $[f] \in X/X_n$ can be identified with an n-tuple $(a_1, \ldots, a_n)$.

Exercise 7.

What is the dimension of the quotient space c/c_0?

From the definition of quotient norm of $x + M$, we can prove the following result.

Proposition 8.

For every $x + M$ and $\varepsilon > 0$, there exists $z \in M$ such that $\|x + z\| < \|x + M\| + \varepsilon$. (OR) For every $x + M$ and $\varepsilon > 0$, there exists x' in the coset $x + M$ such that $x + M = x' + M$ and $\|x'\| < \|x + M\| + \varepsilon$.

Proposition 9.

If M is a finite dimensional subspace of X, then $\|x+M\|$ is attained at some point $y \in x+M$. (OR) Let Y be a finite dimensional subspace of a normed space X. Then for each $x \in X$, there is an element y_0 of Y such that $d(x,Y) = \|x - y_0\|$.

The existence of y_0 is not necessarily unique.

Example 10.

Let $n_0 \in \mathbb{N}$ be fixed. Let $E = \langle \{e_1, \ldots, e_{n_0}\rangle$ be a subset of ℓ_∞ and $x = e_{n_0+1} \in \ell_\infty$. Then $d(x,E) = 1$ and $\|x-y\|_\infty = 1$ for all $y = (\alpha_1, \ldots, \alpha_n, 0, 0, \ldots) \in E$ with $|\alpha_i| \leq 1$.

Results

Proposition 11.

Let M be a closed subspace of a normed space X. Then $(x_n + M)$ converges to $x + M$ iff there is a sequence (y_n) in M such that $x_n + y_n$ converges to x in X.

Three-space property

Definition 12.

Let M be a closed subspace of a normed space X. A property P is a **three-space property** *if two of the spaces X, M, X/M have the property P, then the third must also have the property P.*

Theorem 13.

Finite-dimensionality is a three space property.

Theorem 14.

Completeness is a three space property.

Separable spaces

Definition 15.

A normed space X is a **separable space** *if X has a countable dense subset D. That is, if there exists a countable set D of X such that for every $x \in X$ and $r > 0$, there exists $y \in D$ such that $\|x - y\| < r$.*

We denote $\mathbb{K}_{\mathbb{Q}}$ the set of rationals when $\mathbb{K} = \mathbb{R}$ or the set of complex numbers with rational real and imaginary parts when $\mathbb{K} = \mathbb{C}$.

Theorem 16.

Separability is a three space property.

Theorem 17.

Reflexivity is a three space property.

Dunford-Pettis property is not a three-space property

Definition 18.

A Banach space X is said to have **Dunford-Pettis property** *if any weakly compact operator $T : X \to Y$ transforms weakly compact sets of X into relatively compact sets of Y.*

Jesus M. F. Castillo and Manuel Gonzalez have proved in 1993 that the **Dunford-Pettis property is not a three-space property**.

Example in Algebra

$S_3/A_3 = S_2$ is commutative and A_3 is commutative but S_3 is not commutative.

References

- Jesus M.F. Castillo and Manuel Gonzalez, *Three-space Problems in Banach Space Theory*, Springer, 1997.
- Robert E. Megginson, *An Introduction to Banach Space Theory*, Springer, 1998.